On the Church-Rosser Property for the direct sum of Term Rewriting Systems(Lambda Calculus and Computer Science Theory)

TOYAMA, Yoshihito

数理解析研究所講究録
515巻 1984年 110-133

Lambda Calculus and Computer
Science Theory Symposium
in Kyoto Aug.1983

On the Church-Rosser property for the direct sum of Term Rewriting Systems

Yoshihito TOYAMA

Musashino Electrical Communication Laboratory, N.T.T.
Musashino-shi, 180 Japan

Abstract

The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that if two term rewriting systems both have the Church-Rosser property respectively then the direct sum of these systems also has this property.

1. Introduction

We consider the property of the direct sum system $R_1 \oplus R_2$ obtained from two term rewriting systems R_1 and R_2 [3]. The first study on the direct sum system was conducted by Klop in [3] in order to consider the Church-Rosser property for combinatory reduction systems having nonlinear rewriting rules. He showed that if R_1 is a regular, i.e., linear and non-ambiguous, system and R_2 has only a nonlinear rewriting rule $D(x,x) \triangleright x$, then the direct sum $R_1 \oplus R_2$ has the Church-Rosser property. He also showed in the same manner that if R_2 is a nonlinear system, i.e.,

$$\begin{aligned} &if(T,x,y) \triangleright x, \\ &if(F,x,y) \triangleright y, \\ &if(z,x,x) \triangleright x, \end{aligned}$$

then the direct sum $R_1 \oplus R_2$ also has the Church-Rosser property. This result gave a positive answer for the open problem suggested by O'Donnell [4].

Klop's work was done on combinatory reduction systems. Considering his work from the viewpoint of term rewriting systems [2], R_1 and R_2 are limited by the following structures: R_1 is a nonoverlapping linear system, and R_2 is a nonlinear system having specific rules such as $D(x,x) \triangleright x$. From Klop's work, we consider the conjecture that this limitation can be removed from R_1, R_2 in the framework of term rewriting systems, i.e., the direct sum of R_1 and R_2, independent of their structures such as linearity and ambiguity, always preserves their Church-Rosser property. In this paper we shall prove this conjecture:

> For any two term rewriting systems R_1 and R_2, R_1 and R_2 have the Church-Rosser property iff $R_1 \oplus R_2$ has this property.

2. Notations and Definitions

We explain notions of reduction systems and term rewriting systems, and give definitions for the following sections. We start from abstract reduction systems.

2.1. Reduction Systems

A reduction system is a structure $R=\langle A, \rightarrow \rangle$ consisting of some object set A and some binary relation $\rightarrow$ on A, called a reduction relation. The identity of elements of A (or the syntactical equality) is denoted by $\equiv$. $\xrightarrow{*}$ is the transitive reflexive closure of $\rightarrow$, $\xrightarrow{\equiv}$ is the reflexive closure of $\rightarrow$ and $=$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$). If $x \in A$ is minimal with respect to $\rightarrow$, i.e., $\neg \exists y \in A[x \rightarrow y]$, then we say that x is a normal form, and let $NF_{\rightarrow}$ or NF be the set of normal forms. If $x \xrightarrow{*} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition. $R=\langle A, \rightarrow \rangle$ is strongly normalizing (denoted by $SN(R)$ or $SN(\rightarrow)$) iff every reduction in R terminates, i.e., there is no infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots$.

Definition. $R=\langle A,\rightarrow\rangle$ has the Church-Rosser property, or Church-Rosser, (denoted by CR(R)) iff

$$\forall x,y,z\in A[x\xrightarrow{*}y\wedge x\xrightarrow{*}z \Rightarrow \exists w\in A, y\xrightarrow{*}w\wedge z\xrightarrow{*}w].$$

We express this property with the diagram in Figure 1. In this sort of diagram, dashed arrows denote (existential) reductions depending on the (universal) reductions shown by full arrows.

$$\begin{tikzcd}
x \arrow[r, "*"] \arrow[d, "*"'] & z \arrow[d, dashed, "*"] \\
y \arrow[r, dashed, "*"'] & w
\end{tikzcd}$$

Figure 1

The following properties are well known in [1][2].

Properties. Let CR(R), then,

(1) the normal form of any element, if it exists, is unique,

(2) $\forall x,y\in A[x=y \Rightarrow \exists w\in A, x\xrightarrow{*}w\wedge y\xrightarrow{*}w]$.

2.2. Term Rewriting Systems

Next, we will explain term rewriting systems that are reduction systems having a terms set as a object set A.

Let V be a set of variable symbols denoted by x,y,z,..., and F a set of function symbols denoted by f,g,h,..., where $F\cap V=\emptyset$. An arity function ρ is a mapping from F to natural number $\mathbb{N}$, and if $\rho(f)=n$ then f is called an n-ary function symbol. In particular, a 0-ary function symbol is called a constant.

The set T(F) of terms on a function symbol set F is inductively defined as follows:

(1) $x\in T(F)$ if $x\in V$,

(2) $f \in T(F)$ if $f \in F$ and $\rho(f)=0$,

(3) $f(M_1,\dots M_n) \in T(F)$ if $f \in F$, $\rho(f)=n>0$, and $M_1,\dots,M_n \in T(F)$.

We use T for T(F) when F is clear in the context.

A substitution θ is a mapping from a term set T to T such that;

(1) $\theta(f) \equiv f$ if $f \in F$ and $\rho(f)=0$,

(2) $\theta(f(M_1,\dots,M_n)) \equiv f(\theta(M_1),\dots,\theta(M_n))$ if $f(M_1,\dots,M_n) \in T$.

Thus, for term M, $\theta(M)$ is determined by its values on the variable symbols occurring in M. Following common usage, we write this as $M\theta$ instead of $\theta(M)$.

Consider an extra constant $\square$ called a hole and the set $T(F \cup \{\square\})$. Then $C \in T(F \cup \{\square\})$ is called the context on F. We use the notation C[,...,] for the context containing n holes ($n \geq 0$), and if $N_1,\dots,N_n \in T(F)$ then $C[N_1,\dots,N_n]$ denote the result of placing $N_1,\dots,N_n$ in the holes of C[,...,] from left to right. In particular, C[ ] denotes a context containing precisely one hole.

N is called a subterm of M if $M \equiv C[N]$. Let N be a subterm occurrence of M, then, we write $N \subset M$, and if $N \not\equiv M$ then write $N \subsetneq M$. If N_1 and N_2 are subterm occurrences of M having no common symbol occurrences (i.e., $M \equiv C[N_1,N_2]$), then N_1,N_2 are called disjoint (denoted by $N_1 \perp N_2$).

A rewriting rule on T is a binary relation $\triangleright$ on T, written as $M_l \triangleright M_r$ for $\langle M_l,M_r \rangle \in \triangleright$, such that if $M_l \triangleright M_r$ then any variable in M_r also occurs in M_l. A $\rightarrow$redex, or redex, is a term $M_l\theta$ where $M_l \triangleright M_r$, and in this case $M_r\theta$ is called a $\rightarrow$contractum, or contractum, of $M_l\theta$. The rewriting rule $\triangleright$ on T defines a reduction relation $\rightarrow$ on T as follows:

$$M \rightarrow N \text{ iff } M \equiv C[M_l\theta],\ N \equiv C[M_r\theta] \text{ and } M_l \triangleright M_r$$
$$\text{for some } M_l,\ M_r,\ C[\ ],\ \text{and } \theta.$$

When we want to specify the redex occurrence $A \equiv M_l\theta$ of M in this reduction, write $M \xrightarrow{A} N$.

Definition. A term rewriting system R on T is a reduction system $R=\langle T,\rightarrow \rangle$ such that the reduction relation $\rightarrow$ is defined by a rewriting rule on T. If R has $M_l \triangleright M_r$, then we write $M_l \triangleright M_r \in R$.

If every variable in term M occurs only once, then M is called linear. We say that R is linear iff $\forall M \triangleright N \in R$, M is linear. R is called nonlinear if R is not linear.

Note that in this paper we have no limitation of R, thus, R may have nonlinear and ambiguous (i.e., overlapping) rewriting rules [2][3].

2.3. Direct Sum Systems

Let F_1, F_2 be disjoint sets of function symbols (i.e., $F_1 \cap F_2 = \emptyset$), then term rewriting systems R_1 on $T(F_1)$ and R_2 on $T(F_2)$ are called disjoint. Consider disjoint systems R_1, R_2 having rewriting rules $\triangleright_1$, $\triangleright_2$, respectively, then the direct sum system $R_1 \oplus R_2$ is the term rewriting system on $T(F_1 \cup F_2)$ having the rewriting rule $\triangleright_1 \cup \triangleright_2$. If R_1, R_2 are term rewriting systems not satisfying the disjoint requirement for function symbols, then we take isomorphic copies R_1', R_2' by replacing each function symbol f of F_i by f^i $(i=1,2)$, and use $R_1' \oplus R_2'$ instead of $R_1 \oplus R_2$. For this reason, considering the direct sum $R_1 \oplus R_2$, we may assume that R_1, R_2 are always disjoint, i.e., $F_1 \cap F_2 = \emptyset$.

In this paper we use the following notations: $R_1 = \langle T(F_1), \underset{1}{\rightarrow} \rangle$, $R_2 = \langle T(F_2), \underset{2}{\rightarrow} \rangle$ and $R_1 \oplus R_2 = \langle T(F_1 \cup F_2), \rightarrow \rangle$ where $F_1 \cap F_2 = \emptyset$, $CR(R_i)$ $(i=1,2)$. Note that in the following sections the notation $\rightarrow$ represents the reduction relation on $R_1 \oplus R_2$.

Definition. A root is a mapping from $T(F_1 \cup F_2)$ to $F_1 \cup F_2 \cup V$ as follows: For $M \in T(F_1 \cup F_2)$,

$$root(M) = \begin{cases} f \ldots \text{if } M \equiv f(M_1, \ldots, M_n), \\ M \ldots \text{if } M \text{ is a constant or a variable.} \end{cases}$$

Definition. Let $M \equiv C[B_1, \ldots, B_n] \in T(F_1 \cup F_2)$. Then write $M \equiv C[\![B_1, \ldots, B_n]\!]$ if $C[\ ,\ldots,\ ]$ is a context on F_a and $\forall i, root(B_i) \in F_b$ $(a,b \in \{1,2\}$ and $a \neq b)$. Then the set $Part(M)$ of the parted terms of $M \in T(F_1 \cup F_2)$ is inductively defined as follows:

(1) $Part(M) = \{M\}$ if $M \in T(F_a)$ $(a=1$ or $2)$,

(2) $Part(M) = \bigcup_i Part(B_i) \cup \{M\}$ if $M \equiv C[\![B_1, \ldots, B_n]\!]$ $(n>0)$.

Definition. For a term $M \in T(F_1 \cup F_2)$, the rank $r(M)$ of layers of contexts on F_1 and F_2 in M is inductively defined as follows:

(1) $r(M)=1$ if $M \in T(F_a)$ $(a=1$ or $2)$,

(2) $r(M) = \max_i \{r(B_i)\} + 1$ if $M \equiv C[\![B_1, \ldots, B_n]\!]$ $(n>0)$.

Lemma 2.1. If $M \rightarrow N$ then $r(M) \geq r(N)$.

Proof. It is easily obtained from the definitions of the direct sum $R_1 \oplus R_2$. $\square$

3. Preserved Systems

A term $M \in T(F_1 \cup F_2)$ has a layer structure of contexts on F_1 and F_2, and this structure is modified through a reduction process in a direct sum system $R_1 \oplus R_2$ on $T(F_1 \cup F_2)$. If a reduction $M \to N$ results in the disappearance of some layer between two layers in the term M, then, by putting together two layers, the new layer structure appears in the term N. If no middle layer disappears as a result of any reduction, then we say that the layer structure is preserved in the direct sum system. In this section we will show that if two term rewriting systems have Church-Rosser, then their direct sum having the preserved layer structure also has Church-Rosser. Using this result, we will prove our conjecture in section 4.

The set of terms reduced from a term M by a reduction relation $\to$ is denoted by $G_{\to}(M)=\{N : M \xrightarrow{*} N\}$.

Definition. A term M is root preserved (denoted by r-Pre(M)) iff $root(M) \in F_a \Rightarrow \forall N \in G_{\to}(M), root(N) \in F_a$, where $a \in \{1,2\}$.

Definition. A term $M \equiv C[\![B_1,\ldots,B_n]\!]$ $(n \geq 0)$ is preserved iff M satisfies two conditions;

(1) r-Pre(M),

(2) $\forall i, B_i$ is preserved.

We write Pre(M) when M is preserved. Note that, by the definition, if Pre(M), then $\forall N \in G_{\to}(M)$, Pre(N).

Let $M \xrightarrow{A} N$ and $M \equiv C[\![B_1,\ldots,B_n]\!]$. If the redex occurrence A occurs in some B_j, then we write $M \xrightarrow[i]{} N$, otherwise $M \xrightarrow[o]{} N$. $\xrightarrow[i]{}$ and $\xrightarrow[o]{}$ are called an inner and an outer reduction respectively.

Lemma 3.1. Let Pre(M) and $M \equiv C[\![B_1,\ldots,B_n]\!]$. Then,

(1) $M \xrightarrow[i]{} N \Rightarrow N \equiv C[\![C_1,\ldots,C_n]\!]$ where $\forall i, B_i \xrightarrow{\equiv} C_i$,

(2) $M \xrightarrow[o]{} N \Rightarrow N \equiv C'[\![B_{i_1},\ldots,B_{i_p}]\!]$ $(1 \leq i_j \leq n)$
where $C[\ ,\ldots,\ ]$ and $C'[\ ,\ldots,\ ]$ are contexts on the same set F_a (a=1 or 2).

Proof. It is immediately proved from Pre(M) and the definition of $\xrightarrow[i]{}, \xrightarrow[o]{}$. □

We consider the sequences of terms; $\alpha = \langle A_1,\ldots,A_n \rangle$, $\beta = \langle B_1,\ldots,B_n \rangle$

where $A_i, B_i \in T$. Then, we write $\alpha \propto \beta$ iff $\forall i,j[A_i \equiv A_j \Rightarrow B_i \equiv B_j]$. We define $\alpha \xrightarrow{*} \beta$ by $\forall i, A_i \xrightarrow{*} B_i$.

We extend the above notations to terms. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $N \equiv C[\![B_1, \ldots, B_n]\!]$, $\alpha = \langle A_1, \ldots, A_n \rangle$, $\beta = \langle B_1, \ldots, B_n \rangle$. Then write $M \propto N$ if $\alpha \propto \beta$.

Lemma 3.2. Let Pre(M), $M \propto N$. If $M \xrightarrow[o]{} M'$, then $\exists N', N \xrightarrow[o]{} N' \wedge M' \propto N'$.

Proof. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $N \equiv C[\![B_1, \ldots, B_n]\!]$. Then the left side of the rewriting rule used in $M \xrightarrow[o]{} M'$ occurs in context C[,...,]. Since $M \propto N$ we can apply this rule to N in the same way, and get $N \rightarrow N'$. By Lemma 3.1(2), it is clear that $M' \propto N'$. $\square$

Lemma 3.3. Let Pre(M), $M \xrightarrow[o]{} P$, $M \xrightarrow[i]{*} N$, $M \propto N$. Then there is a term Q satisfying the diagram in Figure 2, i.e., $\forall M,N,P\ T[M \xrightarrow[i]{*} N \wedge M \xrightarrow[o]{} P \wedge M \propto N \Rightarrow \exists Q \in T, N \xrightarrow[o]{} Q \wedge P \xrightarrow[i]{*} Q \wedge P \propto Q]$.

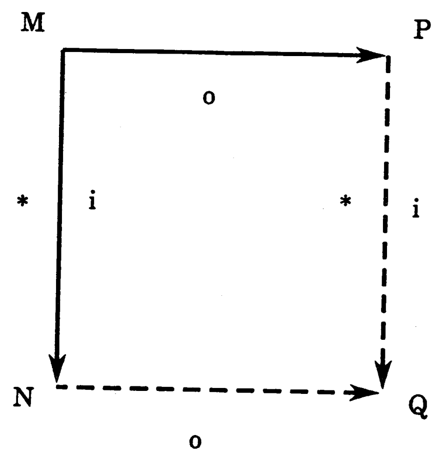

Figure 2

Proof. By Lemma 3.2 we get a term Q such that $P \propto Q$ and $N \xrightarrow[o]{} Q$. Using $M \xrightarrow[o]{} P$, $M \xrightarrow[i]{*} N$ and Lemma 3.1(1), (2), we obtain $P \xrightarrow[i]{*} Q$. $\square$

Lemma 3.4. Let Pre(M), $M \xrightarrow[i]{*} N$, $M \xrightarrow[o]{*} P$, $M \propto N$. Then one get a term Q satisfying Figure 3.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "*", "o"'] \arrow[d, "*"', "i"] & P \arrow[d, dashed, "*"', "i"] \\
N \arrow[r, dashed, "*", "o"'] & Q
\end{tikzcd}
$$

Figure 3

Proof. Using lemma 3.3, the diagram in Figure 4 can be made. □

$$
\begin{tikzcd}[column sep=large, row sep=huge]
M \arrow[r, "o"'] \arrow[d, "*"', "i"] & \cdot \arrow[r, "o"'] \arrow[d, dashed, "*"', "i"] & \cdot \arrow[r, "o"'] \arrow[d, dashed, "*"', "i"] & \cdot \arrow[r, "o"'] \arrow[d, dashed, "*"', "i"] & P \arrow[d, dashed, "*"', "i"] \\
N \arrow[r, dashed, "o"'] & \cdot \arrow[r, dashed, "o"'] & \cdot \arrow[r, dashed, "o"'] & \cdot \arrow[r, dashed, "o"'] & Q
\end{tikzcd}
$$

Figure 4

We define the local Church-Rosser property at a term M.

Definition. Let $R=\langle T,\rightarrow\rangle$ be a reduction system and let $M\in T$. Then M is Church-Rosser for $\rightarrow$ (denoted by $CR_{\rightarrow}(M)$ or $CR(M)$) iff $\forall N,P\in T[M\xrightarrow{*}N\wedge M\xrightarrow{*}P \Rightarrow \exists Q\in T, N\xrightarrow{*}Q\wedge P\xrightarrow{*}Q]$. Note that $\forall M\in T, CR(M)$ iff $CR(R)$.

We define $M\downarrow N$ by $\exists Q\in T, M\xrightarrow{*}Q\wedge N\xrightarrow{*}Q$.

Lemma 3.5. Let $\alpha = \langle A_1, \ldots, A_n \rangle$ and $\forall i, CR(A_i)$. Then $\exists \beta = \langle B_1, \ldots, B_n \rangle\ [\alpha \xrightarrow{*} \beta \wedge \forall i,j [A_i \downarrow A_j \Rightarrow B_i \equiv B_j]]$.

Proof. Using $CR(A_k)$, it can be shown that $A_i \downarrow A_k \wedge A_k \downarrow A_j \Rightarrow A_i \downarrow A_j$. Hence $\downarrow$ is an equivalence relation and partitions $\{A_1, \ldots, A_n\}$ in the equivalence class $C_1, \ldots, C_m$. Using the Church-Rosser for each A_i, we can take a term B_p for each equivalence class $C_p = \{A_{p1}, \ldots, A_{pq}\}$ as the diagram in Figure 5. Take $B_{p1} \equiv, \ldots, \equiv B_{pq} \equiv B_p$. $\square$

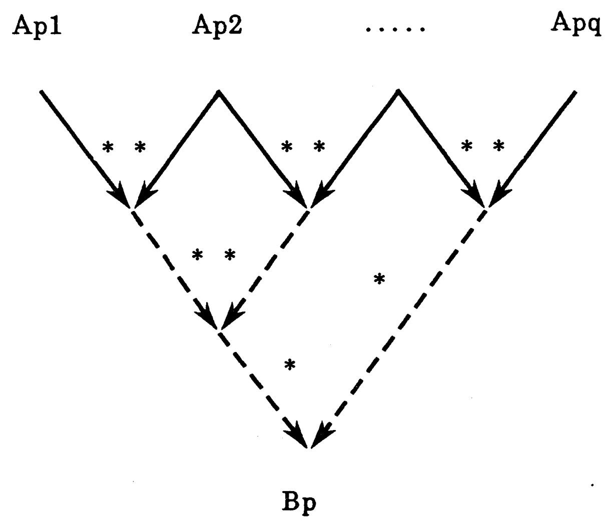

Figure 5

Lemma 3.6. Let $\alpha = \langle A_1, \ldots, A_n \rangle \xrightarrow{*} \beta = \langle B_1, \ldots, B_n \rangle$ and $\forall i, CR(A_i)$. Then $A_i \downarrow A_j$ iff $B_i \downarrow B_j$.

Proof. By the Church-Rosser for each A_i, it is obvious. $\square$

Lemma 3.7. Let $\alpha = \langle A_1, \ldots, A_n \rangle$, $\forall i, CR(A_i)$, and $\alpha \xrightarrow{*} \beta$, $\alpha \xrightarrow{*} \gamma$. Then we can get δ satisfying Figure 6, where $\beta \propto \delta$ and $\gamma \propto \delta$.

$$
\begin{tikzcd}
\alpha \arrow[r, "*"] \arrow[d, "*"'] & \gamma \arrow[d, dashed, "*"'] \\
\beta \arrow[r, dashed, "*"] & \delta
\end{tikzcd}
$$

Figure 6

Proof. Let $\beta = \langle B_1, \ldots, B_n \rangle$, $\gamma = \langle C_1, \ldots, C_n \rangle$. By $\forall i, CR(A_i)$, we have a term $\delta' = \langle D_1', \ldots, D_n' \rangle$ such that $\beta \xrightarrow{*} \delta'$ and $\gamma \xrightarrow{*} \delta'$. Using Lemma 3.5 for δ', we get $\delta = \langle D_1, \ldots, D_n \rangle$ such that $\delta \xrightarrow{*} \delta'$ and $D_i' \downarrow D_j' \Rightarrow D_i \equiv D_j$. Then, by Lemma 3.6, $A_i \downarrow A_j \Leftrightarrow D_i' \downarrow D_j'$, hence $A_i \downarrow A_j \Rightarrow D_i \equiv D_j$. Next we show $\beta \propto \delta$. If $B_i \equiv B_j$, then $A_i \downarrow A_i$, and, thus $D_i \equiv D_j$, hence $\beta \propto \delta$. Similarly we can prove $\gamma \propto \delta$. $\square$

Lemma 3.8. Let $M \equiv C[\![A_1, \ldots, A_n]\!]$, $Pre(M)$, $\forall i, CR(A_i)$. Then we have the diagram in Figure 7, where $N \propto Q$, $P \propto Q$.

$$
\begin{tikzcd}
M \arrow[r, "*", "i"'] \arrow[d, "*"', "i"] & P \arrow[d, dashed, "*"', "i"] \\
N \arrow[r, dashed, "*", "i"'] & Q
\end{tikzcd}
$$

Figure 7

Proof. Since Pre(M), we get $N \equiv C[\![B_1,\ldots,B_n]\!]$, $P \equiv C[\![C_1,\ldots,C_n]\!]$, where $\alpha=\langle A_1,\ldots,A_n\rangle \xrightarrow{*} \beta=\langle B_1,\ldots,B_n\rangle$, $\alpha=\langle A_1,\ldots,A_n\rangle \xrightarrow{*} \gamma=\langle C_1,\ldots,C_n\rangle$. Using Lemma 3.7, we can get $\delta=\langle D_1,\ldots,D_n\rangle$ such that $\beta \xrightarrow{*} \delta$, $\gamma \xrightarrow{*} \delta$, $\beta \propto \delta$ and $\gamma \propto \delta$. Therefore take $Q \equiv C[\![D_1,\ldots,D_n]\!]$. □

Lemma 3.9. If Pre(M), then $CR_{\underset{o}{\rightarrow}}(M)$, i.e., M is Church-Rosser for $\underset{o}{\rightarrow}$ (Figure 8).

$$
\begin{tikzcd}
M \arrow[r, "*", "o"'] \arrow[d, "*"', "o"] & P \arrow[d, dashed, "*"', "o"] \\
N \arrow[r, dashed, "*", "o"'] & Q
\end{tikzcd}
$$

Figure 8

Proof. Let $root(M) \in F_a$ (a=1 or 2). Then, since Pre(M), the outermost part of any term is always a context on F_a. Thus $\underset{o}{\rightarrow}$ is determined by only R_a. Hence Church-Rosser for $\underset{o}{\rightarrow}$ is obvious by $CR(R_a)$. □

Theorem 3.1. If Pre(M), then CR(M).

Proof. By induction on the rank r(M) of layers in M. The case r(M)=1 is trivial since $M \in T(F_a)$ and $CR(R_a)$ (a=1 or 2), therefore, suppose $M \equiv C[\![A_1,\ldots,A_n]\!]$, r(M)=n>1.

Claim: We obtain the diagram in Figure 9.

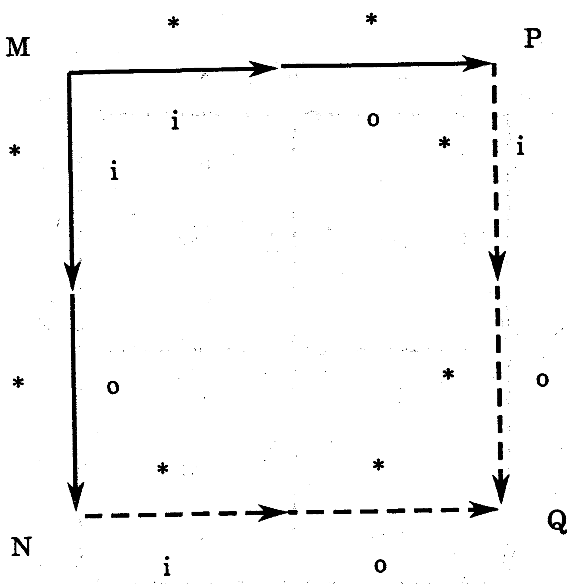

Figure 9

Proof of the claim. By the induction hypothesis, we obtain $\forall i, CR(A_i)$. Using Lemmas 3.8, 3.4 and 3.9 respectively for (1), (2) and (3), we can get the diagram in Figure 10, where $M' \propto Q'$ and $M'' \propto Q'$, hence, we have the claim.

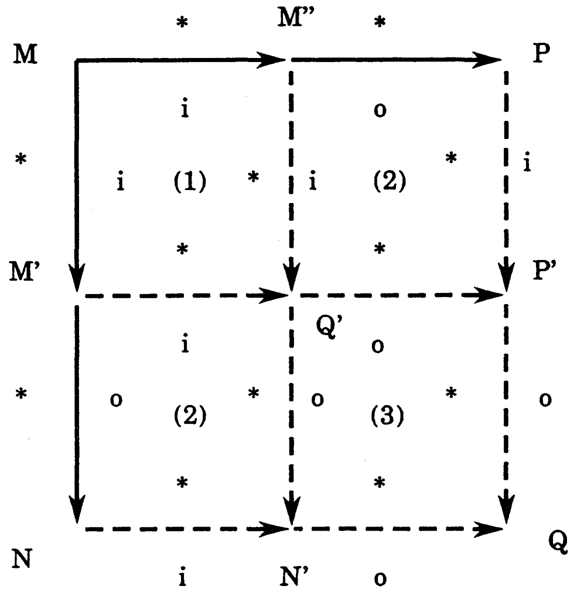

Figure 10

Now we will show CR(M). Note that any reduction $M \xrightarrow{*} M'$ takes the form of $M \xrightarrow[i]{*}\xrightarrow[o]{*} M_1 \xrightarrow[i]{*}\xrightarrow[o]{*} M_2 \xrightarrow[i]{*}\xrightarrow[o]{*} \dots \xrightarrow[i]{*}\xrightarrow[o]{*} M'$. Let $M \xrightarrow{*} N$, $M \xrightarrow{*} P$. By splitting $\xrightarrow{*}$ into $\xrightarrow[i]{*}\xrightarrow[o]{*}$ and using the claim, one can draw the diagram in Figure 11. Hence CR(M). □

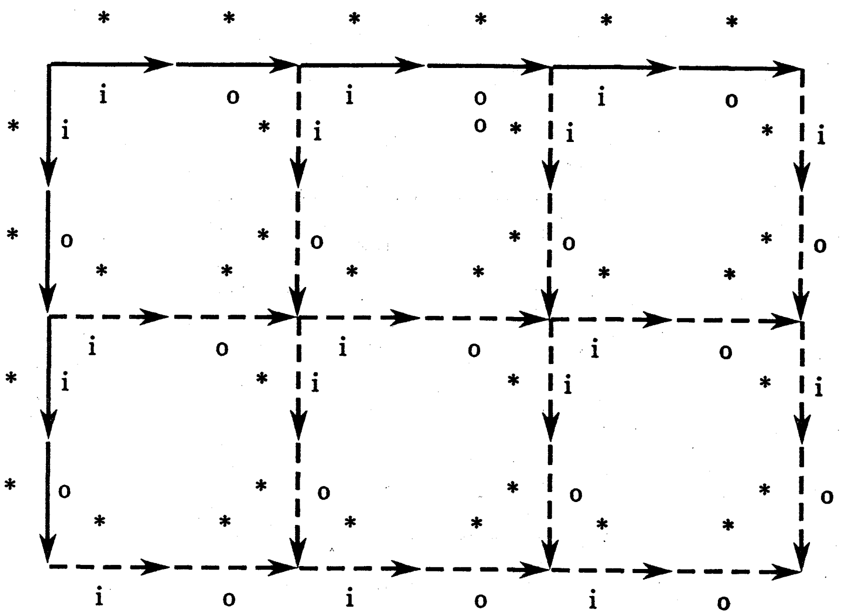

Figure 11

Let $M \xrightarrow{A} N$ where A is a redex occurrence. Then write $M \xrightarrow[p]{} N$ if A occurs in a preserved subterm of M, otherwise write $M \xrightarrow[np]{} N$.

Theorem 3.2. Let $M \equiv C[\![A_1,\ldots,A_n]\!]$, $\forall i, Pre(A_i)$. Then CR(M).

Proof. If Pre(M), immediate by Theorem 3.1. Hence, suppose $\neg$Pre(M). Then one can prove the diagrams (1), (2) and (3) in Figure 12, where $M \propto N$ in (1) and $N \propto Q, P \propto Q$ in (2), in the same way as for Lemmas 3.4, 3.8 and 3.9, respectively, by replacing $\xrightarrow[i]{}$, $\xrightarrow[o]{}$ with $\xrightarrow[p]{}$, $\xrightarrow[np]{}$. Using an analogy to the proof in Theorem 3.1, first, one can obtain the diagram in Figure 13 from the diagrams (1), (2), (3) in Figure 12, and second, splitting $\xrightarrow{*}$ into $\xrightarrow[p]{*}\xrightarrow[np]{*}$, one can show CR(M). □

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
M \arrow[r, "*", "np"'] \arrow[d, "*"', "p"] \arrow[dr, phantom, "(1)"] & P \arrow[d, dashed, "*"', "p"] \\
N \arrow[r, dashed, "*", "np"'] & Q
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
M \arrow[r, "*", "p"'] \arrow[d, "*"', "p"] \arrow[dr, phantom, "(2)"] & P \arrow[d, dashed, "*"', "p"] \\
N \arrow[r, dashed, "*", "p"'] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge, column sep=huge]
M \arrow[r, "*", "np"'] \arrow[d, "*"', "np"] \arrow[dr, phantom, "(3)"] & P \arrow[d, dashed, "*"', "np"] \\
N \arrow[r, dashed, "*", "np"'] & Q
\end{tikzcd}
$$

Figure 12

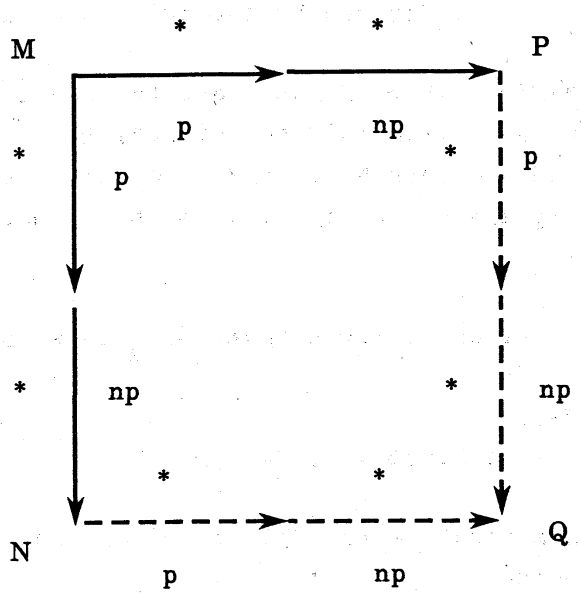

Figure 13

Note: Though $\neg$Pre(M), the above proof is similar to the proof in Theorem 3.1 in which we supposed Pre(M). This analogy come from the fact that in Theorem 3.2 a non-preserved context in a term M only occurs at the outermost part of layer structure. However, if some non-preserved context occurs in the middle part, then one cannot prove CR(M) by the analogous method to Theorem 3.1. In the next section we shall consider this case.

4. The Church-Rosser property for the Direct Sum

In this section we will show that if CR(R_1), CR(R_2), then CR($R_1 \oplus R_2$). This is done by proving CR(M) for any term M by using parallel delete reduction which deletes the layers of the non-preserved contexts occurring in M. First we shall introduce the following delete reduction.

Let a term $M \in T(F_1 \cup F_2)$ not be preserved. Then there is a term $N \in$ Part(M): $N \equiv \tilde{C}[\![B_1,\ldots,B_n]\!]$, $\neg$Pre(N), $\forall i$,Pre(B_i). Since N is not preserved, one has N': $N \xrightarrow{*} N'$, root(N)$\in F_a$, root(N')$\notin F_a$ (a=1 or 2). Then the delete reduction $\underset{d}{\rightarrow}$ is defined by replacing N occurring in M by N' as follows:

$M \underset{d}{\rightarrow} M' \Leftrightarrow M \equiv C[N]$ $M' \equiv C[N']$, where N and N' are the above terms.

Then we say N is $\underset{d}{\rightarrow}$redex. From this definition, $\underset{d}{\rightarrow} \subset \xrightarrow{*}$. Let N_1, N_2 be two different $\underset{d}{\rightarrow}$redex occurrences in M, then it is trivial from the definition that N_1, N_2 are disjoint, i.e., $N_1 \perp N_2$. Note that $M \in NF_{\underset{d}{\rightarrow}}$ iff Pre(M).

Definition. We define the depth $d_M(N)$ of $\underset{d}{\rightarrow}$redex occurrence N in M as follows:

(1) $d_M(N)=1$ if $M \equiv N$,

(2) $d_M(N)=d_{B_i}(N)+1$ if $M \equiv C[\![B_1,\ldots,B_n]\!]$, $N \subset B_i$.

Definition. The maximum depth d(M) of $\underset{d}{\rightarrow}$redex occurrences in M is defined by the following:

(1) $d(M)=0$ if Pre(M),

(2) $d(M)=\max\{d_M(N) :$ N is $\underset{d}{\rightarrow}$redex occurrence in M$\}$ if $\neg$Pre(M).

Note that if $M \rightarrow N$, then $d(M) \geq d(N)$.

Let $N_1,\ldots,N_n$ be all of the $\underset{d}{\rightarrow}$redex occurrences in M having the depth d(M). Note that $N_i \perp N_j$ $(i \neq j)$. Then the parallel delete reduction $\underset{pd}{\rightarrow}$ is defined by replacing each $\underset{d}{\rightarrow}$redex occurrence N_i by N_i' such that $N_i \underset{d}{\rightarrow} N_i$ at one step, or,

$M \underset{pd}{\rightarrow} N \Leftrightarrow M \equiv C[N_1,\ldots,N_n]$, $N \equiv C[N_1',\ldots,N_n']$.

We say that the above $N_1,\ldots,N_n$ are $\underset{pd}{\rightarrow}$redex occurrences. It is clear that $NF_{\underset{pd}{\rightarrow}} = NF_{\underset{d}{\rightarrow}}$. By the definition of parallel delete reduction, one can easily prove that if $M \underset{pd}{\rightarrow} M$ then $d(M)>d(M)$. Hence, every parallel delete reduction terminates, i.e., $SN(\underset{pd}{\rightarrow})$.

Lemma 4.1. Let $M \equiv C[\![A_1,\ldots,A_n]\!] \xrightarrow{M} C'[A_{i_1},\ldots,A_{i_p}]$ where $1 \leq i_p \leq n$, and let $\langle A_1,\ldots,A_n\rangle \propto \langle B_1,\ldots,B_n\rangle$. Then one has a reduction $N \equiv C[B_1,\ldots,B_n] \xrightarrow{N} C'[B_{i_1},\ldots,B_{i_p}]$.

Proof. The left side of the rewriting rule used in the reduction $\xrightarrow{M}$ occurs in context C[,...,], hence, one can apply this rewriting rule to N in the same way as for Lemma 3.2. □

Lemma 4.2. Let $d(M)>1$, $M \equiv C[M_1,\ldots,M_m] \xrightarrow{M} C'[M_{i_1},\ldots,M_{i_p}]$ $(1 \leq i_j \leq m)$, where $M_1,\ldots,M_m$ are all of the $\underset{pd}{\rightarrow}$redex occurrences in M. Let $\langle M_1,\ldots,M_m\rangle \propto \langle M_1',\ldots,M_m'\rangle$. Then one has a reduction

$M' \equiv C[M'_1,\ldots,M'_m] \xrightarrow{M'} C'[M'_{i_1},\ldots,M'_{i_p}]$.

Proof. Let $M \equiv \tilde{C}[\![A_1,\ldots,A_n]\!]$, then $\forall i, \exists j, M_i \subset A_j$, and, thus, by replacing each M_i in A_j with M'_i, to make A'_j, one can get $M' \equiv \tilde{C}[A'_1,\ldots,A'_n]$. Hence, if one show that $\langle A_1,\ldots,A_n\rangle \propto \langle A'_1,\ldots,A'_n\rangle$, then, by using Lemma 4.1, the above Lemma holds. In order to show this, we will prove that if $A_i \equiv A_j$, then $A'_i \equiv A'_j$. Let $A_i \equiv A_j$. If A_i has no $\xrightarrow[pd]{}$redex occurrence in M, then ,by $A_i \equiv A'_i$, it is trivial. Thus, assume A_i to have k (k>0) $\xrightarrow[pd]{}$redex occurrences $M_{r+1},\ldots,M_{r+k}$ in M. Then one can take $A_i \equiv C_i[M_{r+1},\ldots,M_{r+k}]$, $A_j \equiv C_i[M_{s+1},\ldots,M_{s+k}]$, $M_{r+\tilde{i}} \equiv M_{s+\tilde{i}}$ $(1 \le \tilde{i} \le k)$, therefore $A'_i \equiv C'_i[M_{r+1},\ldots,M_{r+k}]$, $A'_j \equiv C_i[M'_{s+1},\ldots,M'_{s+k}]$. By using $\langle M_1,\ldots,M_m\rangle \propto \langle M'_1,\ldots,M'_m\rangle$, one obtains $M'_{r+\tilde{i}} \equiv M'_{s+\tilde{i}}$ $(1 \le \tilde{i} \le k)$. Therefore $A'_i \equiv A'_j$. □

Lemma 4.3. Let $d(M)>1$, $M \equiv C[M_1,\ldots,M_m] \xrightarrow{M} C'[M_{i_1},\ldots,M_{i_p}]$ $(1 \le i_j \le m)$, where $M_1,\ldots,M_m$ are all of the $\xrightarrow[pd]{}$redex occurrences in M. Let $\langle M_1,\ldots,M_m\rangle \xrightarrow{*} \langle M'_1,\ldots,M'_m\rangle$. Then one can obtain a term sequence $\langle M''_1,\ldots,M''_m\rangle$ such that $\langle M'_1,\ldots,M'_m\rangle \xrightarrow{*} \langle M''_1,\ldots,M''_m\rangle$ and $M' \equiv C[M''_1,\ldots,M''_m] \xrightarrow{M'} C'[M''_{i_1},\ldots,M''_{i_p}]$.

Proof. In order to prove the Lemma by using Lemma 4.2, we only need to show a $\langle M''_1,\ldots,M''_m\rangle$ such that $\langle M_1,\ldots,M_m\rangle \propto \langle M''_1,\ldots,M''_m\rangle$. Since $M_1,\ldots,M_m$ are all of the $\xrightarrow[pd]{}$redex occurrences, we get $\forall i, CR(M_i)$ by Theorem 3.2. Therefore we obtain this $\langle M''_1,\ldots,M''_m\rangle$ by Lemma 3.7, taking $\alpha = \langle M_1,\ldots,M_m\rangle$, $\beta = \gamma = \langle M'_1,\ldots,M'_m\rangle$ and $\delta = \langle M''_1,\ldots,M''_m\rangle$. □

Lemma 4.4. Let $M \rightarrow N$, $M \xrightarrow[pd]{} P$, $d(M)=d(N)$. Then one has the diagram in Figure 14. Note that $d(M)>d(S)$.

$$\begin{tikzcd}
M \arrow[r] \arrow[dd, "pd"'] & N \arrow[d, dashed, "pd"] \\
 & Q \arrow[d, dashed, "*"'] \\
P \arrow[r, dashed, "*"] & S
\end{tikzcd}$$

Figure 14

Proof. Let $M \xrightarrow{A} N$. The possible relative positions of the redex occurrence A and all of the $\xrightarrow[p\alpha]{}$redex occurrences in M, say $M_1,\dots,M_m$, are given in the following cases.

Case 1. $\forall i, A \perp M_i$.

Then $M \equiv C[M_1,\dots,M_r,A,M_{r+1},\dots,M_m]$, $N \equiv C[M_1,\dots,M_r,B,M_{r+1},\dots,M_m]$, $P \equiv C[P_1,\dots,P_r,A,P_{r+1},\dots,P_m]$, where $A \xrightarrow{A} B$ and $\forall i, M_i \xrightarrow[\alpha]{\equiv} P_i$. Since all of the $\xrightarrow[p\alpha]{}$redex occurrences in N are also $M_1,\dots,M_m$, we can take $Q \equiv C[P_1,\dots,P_r,B,P_{r+1},\dots,P_m]$. Let $S \equiv Q$, then $P \xrightarrow{*} S$ and $Q \xrightarrow{*} S$.

Case 2. $\exists r, A \subset M_r$.

Then

$M \equiv C[M_1,\dots,M_{r-1},M_r,M_{r+1},\dots,M_m]$, $N \equiv C[M_1,\dots,M_{r-1},N_r,M_{r+1},\dots,M_m]$, $P \equiv C[P_1,\dots,P_{r-1},P_r,P_{r+1},\dots,P_m]$, where $M_r \xrightarrow{A} N_r$ and $\forall i, M_i \xrightarrow[\alpha]{\equiv} P_i$. Since each M_i $(i \neq r)$ is also $\xrightarrow[p\alpha]{}$ redex occurrences in N, by using $\xrightarrow[p\alpha]{}$ for N, one gets $Q \equiv C[P_1,\dots,P_{r-1},Q_r,P_{r+1},\dots,P_m]$, where $N_r \xrightarrow[\alpha]{\equiv} Q_r$, whether N_r is a $\xrightarrow[p\alpha]{}$redex occurrence or not in N. By Theorem 3.2, $CR(M_r)$, therefore there is a term S_r such that $P_r \xrightarrow{*} S_r, Q_r \xrightarrow{*} S_r$. Therefore take $S \equiv C[P_1,\dots,P_{r-1},S_r,P_{r+1},\dots,P_m]$.

Case 3. $\exists j, M_j \subsetneq A$.

Let $M_r,\dots,M_k$ $(r \leq k)$ be all of the $\xrightarrow[p\alpha]{}$redex occurrences in M occurring in A. Then they are also $\xrightarrow[p\alpha]{}$redex occurrences in A. Let $A \equiv D[M_r,\dots,M_k] \xrightarrow{A} D'[M_{i_1},\dots,M_{i_p}]$ $(r \leq i_j \leq k)$.

Then $M \equiv C[M_1,\dots,M_{r-1},D[M_r,\dots,M_k],M_{k+1},\dots,M_m]$, $N \equiv C[M_1,\dots,M_{r-1},D'[M_{i_1},\dots,M_{i_p}],M_{k+1},\dots,M_m]$,

$P \equiv C[P_1,\ldots,P_{r-1},D[P_r,\ldots,P_k],P_{k+1},\ldots,P_m]$, where $\forall i, M_i \xrightarrow[d]{} P_i$. Since $M_1,\ldots,M_{r-1},M_{k+1},\ldots,M_m$ are also $\xrightarrow[pd]{}$redex occurrences in N, whether $M_{i_1},\ldots,M_{i_p}$ are $\xrightarrow[pd]{}$redex occurrences or not in N, one can get $Q \equiv C[P_1,\ldots,P_{r-1},D'[Q_{i_1},\ldots,Q_{i_p}],P_{k+1},\ldots,P_m]$, where $\forall j, M_{i_j} \xrightarrow[d]{\equiv} Q_{i_j}$. Now, by using Lemma 4.3, one can show for the subterm $D[P_r,\ldots,P_k]$ in P that there is a sequence $\langle P'_r,\ldots,P'_k\rangle$ such that $\langle P_r,\ldots,P_k\rangle \xrightarrow{*} \langle P'_r,\ldots,P'_k\rangle$, and $D[P'_r,\ldots,P'_k] \to D'[P'_{i_1},\ldots,P'_{i_p}]$. Take $P' \equiv C[P_1,\ldots,P_{r-1},D'[P'_{i_1},\ldots,P'_{i_p}],P_{k+1},\ldots,P_m]$, then one can have $P \xrightarrow{*} P'$. Since $\forall j, CR(M_{i_j})$, for each j there is S_{i_j} such that $P'_{i_j} \xrightarrow{*} S_{i_j}$, $Q_{i_j} \xrightarrow{*} S_{i_j}$. Therefore take $S \equiv C[P_1,\ldots,P_{r-1},D'[S_{i_1},\ldots,S_{i_p}],P_{k+1},\ldots,P_m]$. □

Lemma 4.5. Let $M \to N$, $M \xrightarrow[pd]{} P$, $d(M) > d(N)$, then one has the diagram in Figure 15. Note that $d(M) > d(S)$.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "pd"'] & N \arrow[d, dashed, "*"'] \\ P \arrow[r, dashed, "*"] & S \end{tikzcd}$$

Figure 15

Proof. One can get a term S in the same way as for case 2 and case 3 in the proof of Lemma 4.4. □

Theorem 4.1. $R_1 \oplus R_2$ has the Church-Rosser property, i.e., the diagram in Figure 16.

$$
\begin{tikzcd}
M \arrow[r, "*"] \arrow[d, "*"'] & P \arrow[d, dashed, "*"] \\
N \arrow[r, dashed, "*"] & Q
\end{tikzcd}
$$

Figure 16

Proof. We will prove CR(M) by induction on the d(M). The case d(M)=0 is trivial from Theorem 3.1. Assume CR(M) for d(M)<n (n>0). Then we will show the following claim.

Claim. One has the diagram in Figure 17 for the case d(M)$\leq$n.

$$
\begin{tikzcd}
M \arrow[r, "*"] \arrow[d, "pd"] & N \arrow[d, dashed, "*"'] \\
P \arrow[r, dashed, "*"] & Q
\end{tikzcd}
$$

Figure 17

Proof of the claim. Let $M \xrightarrow{(m)} N$, where $\xrightarrow{(m)}$ denotes a reduction of m (m>0) steps. Then we prove the claim by induction on m. The case m=0 is trivial. Assume the claim for m-1 (m>0). We will show the diagram for m. Let $M \rightarrow A \xrightarrow{(m-1)} N$.

Case 1. d(M)=d(A). We can obtain the diagram in Figure 18, proving diagram(1) by using Lemma 4.4, diagram(2) by using the induction hypothesis for the claim, and diagram(3) by using the induction hypothesis for the theorem, i.e., CR(B), since d(M)>d(B).

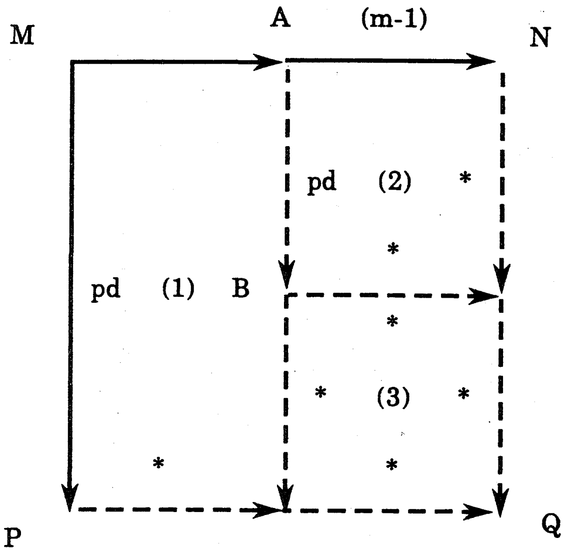

Figure 18

Case 2. d(M)>d(A). We can obtain the diagram in Figure 19, proving diagram(1) by using Lemma 4.5, and diagram(2) by using the induction hypothesis for the theorem, i.e., CR(A).

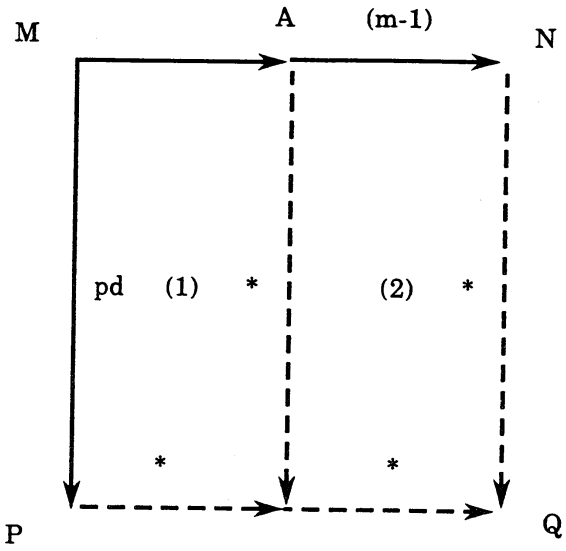

Figure 19

Now we will prove CR(M) for d(M)=n. The diagram in Figure 20 can be obtain, where diagram(1) and diagram(2) are shown by the claim and the induction hypothesis, i.e., CR(A), respectively. □

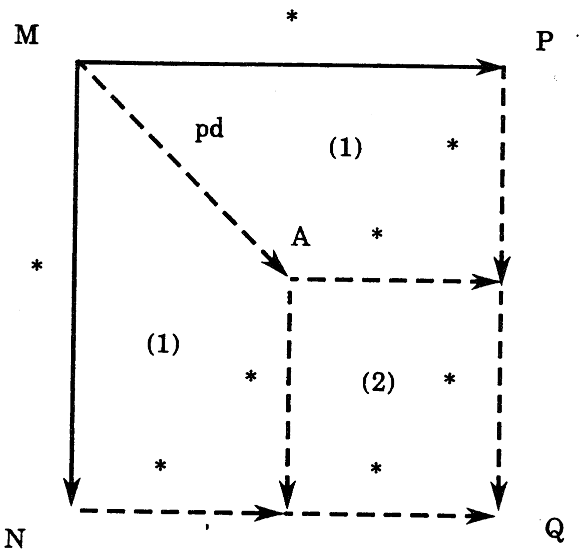

Figure 20

Corollary 4.1. $CR(R_1) \wedge CR(R_2) \Leftrightarrow CR(R_1 \oplus R_2)$.

Proof. $\Leftarrow$ is trivial, and $\Rightarrow$ is proved by Theorem 4.1. $\square$

Acknowledgments

The author is grateful to Hirofumi Katsuno for helpful suggestions.

References

[1] Barendrect,H.P.:" The lambda calculus, its syntax and semantics", North-Holland (1981).

[2] Huet,G.:" Confluent reductions: abstract properties and applications to term rewriting systems", J.ACM, Vol.27 (1980), pp.797-821.

[3] Klop,J.W.:" Combinatory reduction systems", Dissertation, Univ. of Utrecht (1980).

[4] O'Donnell,M.:" Computing in systems described by equations", Lecture Notes in Comput. Sci. Vol.58, Springer-Verlag (1977).

[5] Rosen,B.K.:" Tree-manipulating systems and Church-Rosser theorems", J.ACM, Vol 20 (1973), pp.160-187.

[6] Staples,J.:" Church-Rosser theorem for replacement systems", Lecture Notes in Mathematics 450, Springer (1975), pp.291-307.

[7] Toyama,Y.:" On commutativity of term rewriting systems ", to appear (1983), in Japanese.